PLAIN STRAIN AND PLAIN STRESS

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August 31, 2013

Abstract

Notes on Plain Strain and Plain Stress 2D Finite Elements.

1 PLAIN STRAIN

Strain is In-Plane, but not the stresses. For an isotropic material:

$$\epsilon_{33} = \epsilon_{13} = \epsilon_{23} = 0$$
$$\sigma_{13} = \sigma_{23} = 0$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ 0 \\ 0 \end{bmatrix} = [C] \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 0 \\ \epsilon_{12} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+v)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1-\nu^2 & -\nu(1+\nu) & 0 \\ -\nu(1+\nu) & 1-\nu^2 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

De $\sigma = C\epsilon$ (3D) tengo:

$$\sigma_{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\epsilon_{11}+\epsilon_{22}) + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\epsilon_{33}$$

$\epsilon_{33} = 0$, luego:

$$\sigma_{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}(\epsilon_{11}+\epsilon_{22})$$

2 PLANE STRESS

Stress is In-Plane, but not the strains. For an isotropic material:

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$
$$\epsilon_{13} = \epsilon_{23} = 0$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ 0 \\ \sigma_{12} \\ 0 \\ 0 \end{bmatrix} = [C] \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

De $\sigma = C\epsilon$ (3D) tengo:

$$\epsilon_{33} = -\frac{\nu}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} + \frac{1}{E}\sigma_{33}$$

$\sigma_{33} = 0$, luego:

$$\epsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22})$$

3 FOR NUMERCAL INTEGRATION

When integrating the stiffness matrix:

$$F_{ij} = B_{ij}^T C B_{ij} det(J_{ij})$$

In the case of plain stress or plain strain conditions, we integrate in the r, s plane and assume that the function F is constant through the thickness of the element. The stiffness matrix of the element is therefore:

$$K = \sum_{i,j} t_{ij} \alpha_{ij} F_{ij}$$

where t_{ij} is the thickness of the element at the sampling point (r_i, s_j) ($t_{ij} = 1.0$ in plain strain analysis).